Differentiation and its Applications

Definition

A function $f:(a,b)\to R$ is said to be derivable or differentiable at $x\in(a,b)$ if

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(c-h)-f(c)}{-h}$$

exists. This limit ——— ①

is called the derivative of f w.r.to x. The derivative of f is denoted by f' or $\frac{dy}{dx}$.

Let $c\in(a,b)$ and f be derivable at c, then using the above definition

$$\left.\frac{dy}{dx}\right]_{x=c}=f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad \text{——— (2)}$$

If the limit in (2) exists when $h\to 0^+$ the limit is called the right hand derivative of f at c and is denoted by $f'(c+)$. Similarly we define the left hand derivative at c and denote it by $f'(c-)$.

Thus

$$f'(c+)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h},\ h>0 \quad \text{——— (3)}$$

$$f'(c-)=\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h},\ h>0 \quad \text{——— (4)}$$

It is easily seen that the derivative of $f(x)$ exists at $x=c$ if and only if $f'(c+)$ and $f'(c-)$ both exist and are equal and is denoted by (2) irrespective of sign of h

Exa:

Differentiate (i) x, (ii) $x+3$ (iii) $3x+4$

Sol^n

(i) Let $y = x$

Then $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x) - x}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1$

(ii) Let $y = x+3$

Then $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x+3) - (x+3)}{\delta x}$

$= \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1$

(iii) Let $y = 3x+4$

Then $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{3(x+\delta x)+4 - (3x+4)}{\delta x}$

$= \lim_{\delta x \to 0} \frac{3\delta x}{\delta x} = 3$

Exa: Find the derivative of $\sqrt{x+1}$ at $x=2$.

Sol^n Let $y = \sqrt{x+1}$

By (2), $\left.\frac{dy}{dx}\right]_{at\, x=2} = \lim_{h \to 0} \frac{\sqrt{2+h+1} - \sqrt{2+1}}{h}$

$= \lim_{h \to 0} \left(\frac{\sqrt{3+h} - \sqrt{3}}{h} \right) \frac{\sqrt{3+h}+\sqrt{3}}{\sqrt{3+h}+\sqrt{3}}$

$= \lim_{h \to 0} \frac{3+h-3}{h(\sqrt{3+h}+\sqrt{3})}$

$= \lim_{h \to 0} \frac{1}{\sqrt{3+h}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$

Questions

1. Find the derivative of the following functions using the definition

(i) x^2+1 (ii) $\frac{1}{x}$

(iii) $\frac{1}{3x+2}$ (iv) $\frac{1}{x^2}$

2. Find the derivative of the following functions from definition at the indicated points.

(i) $2x^2+x+1$ at $x=1$

(ii) x^3+2x^2-1 at $x=0$

Tangent line of a graph at a point:

The tangent line to the graph of a function f at the point $P=(x_0, f(x_0))$ is

(i) the line through P with slope $f'(x_0)$ if $f'(x_0)$ exists;

(ii) the line $x=x_0$ if $\lim_{x\to x_0}\left|\frac{f(x)-f(x_0)}{x-x_0}\right| = \infty$

If (i) holds, then clearly the eqn of the required tangent line is:

$$y-f(x_0) = f'(x_0)(x-x_0).$$

Derivative of Some Standard Functions

Exa: If $y=\cos x$ then $\frac{dy}{dx} = -\sin x$

Proof. $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\cos(x+\delta x)-\cos x}{\delta x}$

$$= \lim_{\delta x\to 0}\frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= -\lim_{\delta x\to 0}\sin\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$= \sin x \cdot 1$

$= \sin x$

Questions:

① Find derivatives of the following functions from definition

(i) $3x^2 - \frac{4}{x}$ (ii) $2x + x + \sqrt{x^3}$

(iii) $\sqrt{x^2-1}$

Algebra of derivatives

Let u & v be two derivable functions of x. Then

(i) $(u+v)' = u' + v'$

(ii) $(u-v)' = u' - v'$

(iii) $(uv)' = u'v + v'u$

(iv) $\frac{df}{dx} = \frac{u}{v}$, provided $v(x) \neq 0$.

Ex a: Find $\frac{dy}{dx}$ if $y = x^3 - x^2 + 6$

Solⁿ:

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - x^2 + 6)$$

$$= \frac{dx^3}{dx} - \frac{dx^2}{dx} + \frac{d6}{dx}$$

$$= 3x^2 - 2x + 0$$

$$= 3x^2 - 2x$$

Questions

Differentiate

① $x^8 + x^7$ ② $x^2 + 2x - \sin x + 5$

③ $ax^2 + b\tan x + \ln x^3$ ④ $\tan 2x + \sec 2x$

Derivative of a Composite Function

Let $y = f(u)$ be a differentiable function of u and $u = g(x)$ be a differentiable function of x, so that $y = fog(x)$ is a composite function of x.

Now let δx be a small increment in x and δu, δy be corresponding increments in u and y respectively.

Since $u = g(x)$ is a differentiable function of x it is continuous and hence $\delta u \to 0$ as $\delta x \to 0$.

Now $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

The result

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

is called the chain rule of differentiation.

Questions

Find derivatives of the following functions.

(1) $(x^2+5)^8$ (2) $\sin 5x + \cos 7x$

(3) $\sec(\tan x)$ (4) $\sin^2 x \cos^2 x$

(5) $\ln(\tan x)$

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Derivatives of Inverse Functions

Let f be a differentiable function of x which admits of an inverse function f^{-1}.

Then $\frac{df^{-1}}{dy} = \frac{1}{\left(\frac{df}{dx}\right)}$ provided $\frac{df}{dx} \neq 0$,

Or equivalently $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$, provided $\frac{dy}{dx} \neq 0$. ——(1)

Exa: Find derivative of $\tan^{-1}(\sin^2 x)$

Sol: Let $y = \tan^{-1}(\sin^2 x) = \tan^{-1} u$,

where $u = \sin^2 x$

Then $\frac{dy}{du} = \frac{d}{du}(\tan^{-1} u) = \frac{1}{1+u^2}$

$\frac{du}{dx} = \frac{d}{dx}(\sin^2 x) = 2\sin x \frac{d}{dx}\sin x$

$= 2\sin x\cos x = \sin 2x$

Hence $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{1+u^2}\sin 2x$

$= \frac{\sin 2x}{1+\sin^4 x}$

Questions

(1) $\sin^{-1} 2x$ (2) $\tan^{-1}(\cos\sqrt{x})$

(3) $\sec^{-1}(2x+1)$ (4) $(x\sin^{-1}x)^{15}$

Differentiation by Substitution

Exa: Differentiate $\cos^{-1}(4x^3-3x)$

Sol: Let $y = \cos^{-1}(4x^3-3x)$

Let $x = \cos\theta$ so that $\theta = \cos^{-1} x$

Then $y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$= \cos^{-1}(\cos 3\theta)$

$= 3\theta = 3\cos^{-1} x.$

Hence, $\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}.$

Questions

Differentiate the following functions by proper substitution.

1. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ 2. $\tan^{-1}\frac{2x}{1-x^2}$,

3. $\tan^{-1}\sqrt{\frac{1-t}{1+t}}$ 4. $\cos^{-1}(2t^2-1)$

Differentiation using logarithms:

(i) When a function appears as an exponent of another function we make use of logarithms.

Ex 9: Differentiate $(\sin x)^{\tan x}$.

Soln: Let $y = (\sin x)^{\tan x}$.

Taking logarithms of both sides we have

$\ln y = \ln(\sin x)^{\tan x}$

$= \tan x \ln \sin x$

Differentiating both sides w.r.t. x we obtain

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}\tan x \ln \sin x + \tan x \frac{d}{dx}\ln \sin x$

$= \sec^2 x \ln \sin x + \tan x \frac{\cos x}{\sin x}$

$= \sec^2 x \ln \sin x + 1$

$\Rightarrow \frac{dy}{dx} = y(\sec^2 x \ln \sin x + 1).$

$= (\sin x)^{\tan x} (\sec^2 x \ln \sin x + 1)$

Questions

① x^x ② $x^{\sin x}$

③ $(\log x)^{\tan x}$ (4) $(1+\sqrt{x})^{x^2}$

⑤ $(\tan x)^{\log x^3}$

Differentiation of Implicit Functions

An equation $F(x,y)=0$ in two variables in which x & y are the independent and independent varibles respectively may determine one or more functions. Any such functions as determined from $F(x,y)=0$ Is known as implicit function.

Questions

Find $\frac{dy}{dx}$

① $xy^2 + xy + 1 = 0$ ② $x^2 + 3y^2 = 5$

③ $y^2 \cot x = x^2 \cot y$ ④ $x = y \ln(xy)$

⑤ $y^x = x^{\sin y}$

Exa: Find $\frac{dy}{dx}$ if $y^3 + 3x^2y - 2x = 10$

Sol^n: Differentiating both sides w.r.to x we have

$$\frac{dy^3}{dx} + \frac{d}{dx}(3x^2y) - \frac{d}{dx}(2x) = 0$$

7) $3y^2 \frac{dy}{dx} + 3x^2 \frac{dy}{dx} + 6xy - 2 = 0$

$\Rightarrow 3y^2 \frac{dy}{dx} + 3\left(x^2 \frac{dy}{dx} + 2xy\right) - 2 = 0$

$\Rightarrow (3y^2 + 3x^2) \frac{dy}{dx} + 6xy - 2 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2 - 6xy}{3(x^2+y^2)}$

$\Rightarrow \frac{dy}{dx} = \frac{2(1-3xy)}{3(x^2+y^2)}$

Differentiation of Parametric Functions:

Exa: Find $\frac{dy}{dx}$ if $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$

and $\frac{dy}{dt} = a(\cos t - \cos t - t(-\sin t)) = at\sin t$

Hence $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{at\sin t}{at\cos t} = \tan t$

* Refer book for definition

Questions

Find $\frac{dy}{dx}$

1. $x = a\cos\theta$, $y = a\sin\theta$
2. $x = at^2$; $y = 2at$ at $t = \frac{1}{2}$
3. $x = a\cos^3 t$, $y = a\sin^3 t$ at $t = \frac{\pi}{4}$ *

Differentiation w.r.to a function

Suppose we have two differentiable functions given by $y=f(x)$ and $z=g(x)$. To find derivative of y w.r.to z we regard x as a parameter and find $f'(x)=\frac{dy}{dx}$ and $g'(x)=\frac{dz}{dx}$.

Then

$$\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=\frac{f'(x)}{g'(x)}$$

Exa: Differentiate $\tan^{-1}x$ w.r.to $\cos^{-1}x$.

Sol^n:

Let $y=\tan^{-1}x$ and $z=\cos^{-1}x$.

We have to find $\frac{dy}{dz}$. Now

$$\frac{dy}{dx}=\frac{1}{1+x^2} \quad \text{and} \quad \frac{dz}{dx}=\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=-\frac{\sqrt{1-x^2}}{1+x^2}$$

Questions

Differentiate

1. $\sin x$ w.r.to $\cot x$
2. $\tan^{-1}x$ w.r.to $\tan^{-1}\sqrt{1+x^2}$
3. $\sqrt{x}$ w.r.to x^2

Succesive Differentiation

★ Refer book for definition

Exa: Find y_4 if $y=x^5+4x^3-2x^2+1$

Sol^n: Differentiating succesively we get

(21) $y_1 = \frac{d}{dx} x^5 + 4\frac{d}{dx} x^3 - 2\frac{d}{dx} \cdot x^2 + \frac{d}{dx}(1)$

$= 5x^4 + 12x^2 - 4x.$

$y_2 = 20x^3 + 24x - 4$

$y_3 = 60x^2 + 24.$

$y_4 = 120x.$

Exa: If $y = \cos^4 x$ find y_n.

Sol:

$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}\left\{1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right\}$

$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

$y_n = 0 + \frac{1}{2} 2^n \cos\left(\frac{n\pi}{2} + 2x\right) + \frac{1}{8} \cdot 4^n \cos\left(\frac{n\pi}{2} + 4x\right)$

$= 2^{n-1}\cos\left(\frac{n\pi}{2} + 2x\right) + 2^{2n-3}\cos\left(\frac{n\pi}{2} + 4x\right)$

Questions

(1) Find y_1, y_2, y_3, y_4, in each of the following cases where y is given by

(i) x^7 (ii) $\cos 2x$ (iii) $\sin(3x+1)$

(iv) $\tan x$ (v) $(x^2+1)^3$

Partia Differentiation

Refer book for definition

Exa

If $z = 2x^2y + xy^2 + 5xy$ then find $\frac{\partial z}{\partial x}$ and $\partial z/\partial y$.

Soln:

$$z = 2x^2y + xy^2 + 5xy$$

Taking y as a constant and differentiating w.r.to x, we get

$$\frac{\partial z}{\partial x} = 4xy + y^2 + 5y$$

Taking x as a constant and differentiating w.r.to y, we get

$$\frac{\partial z}{\partial y} = 2x^2 + 2xy + 5x$$

Questions.

1. Find f_x, f_y where $f(x,y)$ is given as follows :-

(i) $\frac{x^2y + y^2x}{x+y}$

(ii) $x^y + y^x$

(iii) $\sin(2x^2 + y^2)$

(iv) $e^{2x}\cos 3y$